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Time-optimal steering of a point mass to a specified position with the required velocity☆

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Abstract

The planar problem of the time-optimal steering of a point mass to the origin of coordinates with a required velocity by means of a force of limited modulus is considered. The optimal control and the optimal time are investigated for different characteristic values of the positional radius vector and arbitrary values of the magnitude and direction of the initial velocity. The results are extended to the case of boundary conditions which are imposed on the magnitude and direction of the initial velocity in the form of inequalities and are transferred to a problem with control for a fixed time.

The model problem considered is of particular interest in flight control mechanics. The use of an optimality criterion in the form of a maximum principle[1] is found to be effective for solving it. The case of an arbitrary dimensionality of the initial and final values of the phase vectors of the system reduces in the situation of a general position to the treatment of the motion in the three-dimensional space formed by the above-mentioned vectors.[2,3]

Note that the equations for the optimal controlled motion can be explicitly integrated analytically in the form of exceedingly bulky expressions using elementary algebraic functions.[4] However, the solution of the boundary value problem of the maximum principle for arbitrary values of the phase vectors at the initial and final instants of time reduces to a system of three transcendental equations in the optimal time and two modified integration constants of the coupled system.[2,3] This makes a qualitative analysis of the solution, based on efficient calculations of the roots of the above-mentioned equations, much more difficult. At the present time, such investigations have been carried out in the case of particular and modified formulations: the classical case of a one-dimensional system (a point on a straight line[1]), a zero value of the final velocity (a “soft landing”[4]). equality of the initial and final velocities,[2] return to the initial point with the required velocity[3] and steering of the system to the origin of coordinates without specifying a velocity.[1,5–7] These situations lead to a planar (two- or one-dimensional) time-optimal problem.

The general case of arbitrary fixed values of the initial position and final velocity is considered below in the case of a two-dimensional problem.

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1. Formulation of the problem

Consider the planar problem of the time-optimal control of a dynamic object by means of a force of limited modulus. It is required that the object be steered to the origin of coordinates with a specified non-zero velocity. The equations of motion, the functional and the constraints in normalized dimensionless variables have the form

$$\dot{x} = \upsilon, \quad x(0) = x_0, \quad x(t_f) = 0; \quad \dot{\upsilon} = u, \quad \upsilon(0) = \upsilon_0, \quad \upsilon(t_f) = \upsilon_f = (-1, 0)$$
$$|u| \le 1, \quad t_f \to \min_u, \quad x, \upsilon \in R^2 \tag{1.1}$$

The more general problem of the magnitudes of the final velocity and of the control action reduces to problem (1.1). The case of an arbitrary final position and the direction of the final velocity reduces to a displacement and rotation of the system of coordinates which are considered. The case of arbitrary dimensionality of the vectors x_0, υ_0 and υ_f, lying in a single plane, is treated in a similar manner. The problem has axial symmetry about the direction of the final velocity υ_f. It was pointed out above that problem (1.1) has been solved[1] for the case of one-dimensional motion (the vectors x_0, υ_0, υ_f are collinear) and for its different modifications.[2–5]

Applying an optimality criterion in the form of the maximum principle,[1] we obtain[1–4]

$$H = (p, \upsilon) + (q, u), \quad \dot{p} = 0, \quad \dot{q} = -p$$
$$p = p_0 = \text{const}, \quad q = -p_0 t + q_0, \quad q_0 = \text{const}$$
$$u(t) = \frac{q}{|q|}, \quad u = \frac{-p_0 t + q_0}{|-p_0 t + q_0|} \tag{1.2}$$

where p and q are variables conjugate to x and υ, and H is the contracted Hamiltonian function.

Systems (1.1) and (1.2) can be completely integrated in terms of elementary functions. Note that the special control conditions, corresponding to the case when $q \equiv 0$, do not satisfy the maximum principle.

We will now normalize the quantity $|q_0|$. We substitute the expression obtained for the control into equation of motion (1.1) and integrate with respect to t. We obtain intermediate expressions which depend on the vector parameters ξ and η and their scalar characteristics ρ and σ:

$$\xi = \frac{p_0}{|q_0|}, \quad \eta = \frac{q_0}{|q_0|}, \quad |\xi| = \rho, \quad |\eta| = 1$$
$$u(t) = \frac{Q(t)}{R(t)}, \quad Q(t) = -\xi t + \eta, \quad R(t) = (\rho^2 t^2 - 2\sigma\rho t + 1)^{1/2}$$
$$\upsilon(t) = \upsilon_0 + \int_0^t \frac{Q(\tau)}{R(\tau)} d\tau = \upsilon_0 + \frac{1}{\rho^2}[-\xi R(\tau) + (\rho\eta - \sigma\xi)V(\tau)]_0^t \tag{1.3}$$
$$V(t) = \ln(\kappa + (1 + \kappa^2)^{1/2}), \quad \kappa = (\rho t - \sigma)(1 - \sigma^2)^{-1/2}$$
$$\sigma = \cos(\xi, \eta), \quad -1 \le \sigma \le 1$$

Hence, the velocity of the object depends explicitly on the time t, the specified initial velocity vector υ_0 and the unknown vectors ξ and η. Using the formula for repeated integration, on the basis of relations (1.3) we obtain the dependence of the radius vector of the position of the point mass on time and on the above-mentioned parameters

$$x(t) = x_0 + \upsilon_0 t + \int_0^t \frac{(t-\tau)Q(\tau)}{R(\tau)} d\tau = x_0 + \upsilon_0 t - \frac{t}{\rho^2}[-\xi R(0) + (\rho\eta - \sigma\xi)V(0)]$$
$$+ \frac{\xi}{2\rho^3}[(\rho\tau + 3\sigma)R(\tau) + (3\sigma^2 - 1)V(\tau)]_0^t - \frac{\eta}{\rho^2}[\sigma V(\tau) + R(\tau)]_0^t$$

We will write the intermediate representations of the motion obtained as functions of "linear" expressions in ξ and η, the coefficients of which depend non-linearly on ρ and σ:

$$\begin{aligned}
v(t) &= v_0 + V_\xi(t)\xi + V_\eta(t)\eta, \quad x(t) = x_0 + v_0 t + X_\xi(t)\xi + X_\eta(t)\eta \\
V_\xi &= -\rho^{-2}(\sigma V + R)|_0^t, \quad V_\eta = \rho^{-1}V|_0^t \\
X_\xi &= \frac{1}{2\rho^3}[(-\rho\tau + 3\sigma)R + (-2\rho\sigma\tau + 3\sigma^2 - 1)V]_0^t + \frac{t}{\rho^2}(1 + \sigma V(0)) \\
X_\eta &= \rho^{-2}[-R + (\rho\tau - \sigma)V]_0^t - \frac{t}{\rho}V(0)
\end{aligned} \tag{1.4}$$

It is obvious that $x(0) = x_0$, $v(0) = v_0$; and we will consider the final conditions (1.4) for $x(t)$ and $v(t)$. Substituting conditions (1.1) at the final point $x(t_f) = 0$, $v(t_f) = (-1, 0)$ into relations (1.4), we obtain a system of four algebraic equations which are "linear" in the vectors ξ and η and non-linear in t_f:

$$-x_0 - v_0 t_f = X_\xi(t_f)\xi + X_\eta(t_f)\eta, \quad v_f - v_0 = V_\xi(t_f)\xi + V_\eta(t_f)\eta \tag{1.5}$$

In the situation of a general position, system (1.5) has a unique solution.[2,3]

We now introduce the unknown vector $\zeta = \xi t_f$ and its modulus $|\zeta| = \mu = \rho t_f$. This substitution is regular since it is known that $t_f > 0$. Taking account of the conditions at the final point, we have a system of four equations in the unknown vectors ζ and η and, also, the optimal time t_f

$$\begin{aligned}
x_0 + v_f t_f &= t_f^2(a_\zeta\zeta + a_\eta\eta), \quad v_0 - v_f = t_f(b_\zeta\zeta + b_\eta\eta) \\
a_\zeta &= -\frac{(\mu + 3\sigma)a + \mu + (3\sigma^2 - 1)b}{2\mu^3}, \quad a_\eta = b_\zeta = \frac{a + \sigma b}{\mu^2}, \quad b_\eta = -\frac{b}{\mu} \\
a &= (\mu^2 - 2\mu\sigma + 1)^{1/2} - 1, \quad b = \ln\left(\frac{\mu - \sigma + (\mu^2 - 2\mu\sigma + 1)^{1/2}}{1 - \sigma}\right)
\end{aligned} \tag{1.6}$$

We note that the coefficients $a_{\zeta,\eta}$ and $b_{\zeta,\eta}$ depend solely on the unknowns μ and σ. The introduction of the parameter μ instead of ρ enables us to separate out the optimal time t_f.

We now reduce the system of vector Eq. (1.6) to three scalar equations by the operation of scalar multiplication

$$\begin{aligned}
(v_0 - v_f)^2 &= t_f^2 f_v^2(\mu, \sigma), \quad x_0^2 = t_f^4 f_x^2(\mu, \sigma) - 2|x_0|ct_f - t_f^2 \\
|x_0||v_0 - v_f|d_1 &= -t_f|v_0 - v_f|d_2 + t_f^3 f_{xv}(\mu, \sigma) \quad (v_f^2 = 1)
\end{aligned} \tag{1.7}$$

According to equalities (1.7), the equations have a polynomial dependence on the unknown t_f and, by means of the first equality, it is easy to eliminate this unknown and to consider a system of two equations (the second and the third) in the unknowns $\mu \geq 0$ and σ, $|\sigma| \leq 1$. The coefficients f_x, f_v, f_{xv} are known functions of μ and σ, and the parameters c, d_1, d_2 have the meaning of the cosines of the angles between the given vectors (x_0, v_f), $(x_0, v_0 - v_f)$ and $(v_0, v_0 - v_f)$ respectively. We have

$$\begin{aligned}
f_x^2 &= a_\zeta^2\mu^2 + 2a_\zeta a_\eta\mu\sigma + a_\eta^2, \quad f_v^2 = b_\zeta^2\mu^2 + 2b_\zeta b_\eta\mu\sigma + b_\eta^2 \\
f_{xv} &= a_\zeta b_\zeta\mu^2 + (b_\zeta a_\eta + b_\eta a_\zeta)\mu\sigma + a_\eta b_\eta \\
c &= \frac{(x_0, v_f)}{|x_0|}, \quad d_1 = \frac{(x_0, v_0 - v_f)}{|x_0||v_0 - v_f|}, \quad d_2 = \frac{(v_f, v_0 - v_f)}{|v_0 - v_f|}
\end{aligned}$$

The special cases when $v_0 = v_f$ and $x_0 = 0$ have been investigated earlier in Refs. 3,4

Hence, after eliminating t_f, according to the first equation of (1.7) we have a system of two equations in the unknowns μ and σ which is solved numerically using Newton's method.[3] The solution of problem (1.1) reduces to solving system

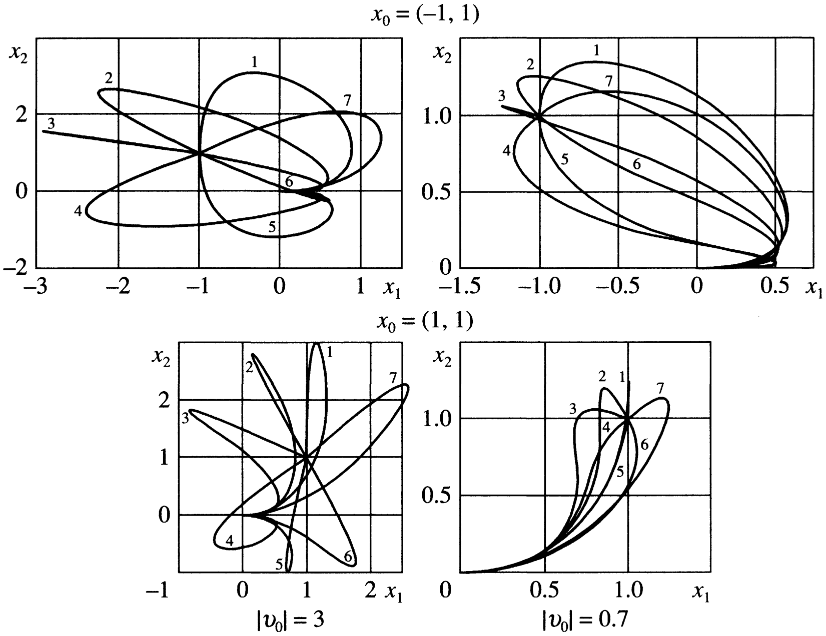

Fig. 1.

Table 1

Curve No.	1	2	3	4	5	6	7	
α/π	0.5	0.75	0.91	−0.8	−0.5	−0.24	0.25	for $x_0 = (-1, 1)$
α/π	0.5	0.65	0.85	−0.8	−0.57	−0.35	0.25	for $x_0 = (1, 1)$

of transcendental equations (1.7) in order to find the parameters μ, σ and t_f and the system of linear equations (1.5) in the vectors ξ and η.

2. Optimal trajectories

Characteristic families of optimal trajectories have been constructed in Fig. 1 for two fixed initial positions: $x_0 = (-1, 1)$ (the upper part of Fig. 1) and $x_0 = (1, 1)$ (the lower part) for fixed values of the modulus of the initial velocity v_0 and different directions of it which are characterized by the angle $\alpha \in (-\pi, \pi)$. The curves correspond to the angles α as shown in Table 1.

It has been established by mathematical modelling that, depending on the directions of the vectors η and v_0 and, also, on the values of $u(t_f)$ and v_f, there are four basic types of trajectories (Fig. 2): Type 1: a motion to the right initially, that is, the angle $\alpha - \pi < \phi < \alpha$ and, then, a bend to the right at the end, that is, the angle $2\pi > \psi > \pi$; type 2: a motion to the left initially (the angle $\alpha + \pi > \phi > \alpha$) and, then, a bend to the right at the end (the angle $2\pi > \psi > \pi$); type 3: a motion to the left initially (the angle $\alpha + \pi > \phi > \alpha$) and, then, a bend to the left at the end (the angle $0 < \psi < \pi$); type 4: a motion to the right initially (the angle $\alpha - \pi < \phi < \alpha$) and, then, a motion to the left at the end (the angle $0 < \psi < \pi$). Here ϕ and ψ are angles which characterize the direction of the control $u(t)$ at the initial and final points respectively. Points at which the vector of the initial velocity v_0 is collinear or anticollinear with the conjugate vector

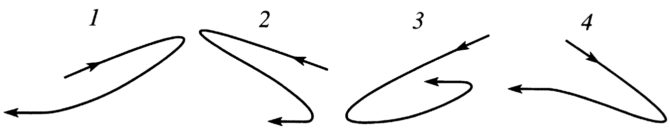

Fig. 2.

η and, also, points at which the vectors v_f and $u(t_f)$ are collinear or anticollinear serve as a transition between these types of trajectories.

3. Constant optimal control

The solution of the following problem is of interest. Suppose the final conditions of the motion of a point mass and the initial position are specified according to the formulation of problem (1.1), and we assume that the initial velocity is unspecified. It is required to find an initial velocity such that the optimal time t_f should be a minimum. This problem was considered in greater detail earlier in Ref. 5. In accordance with the maximum principle, the optimal control will be a constant vector and the trajectory will be parabolic. The optimal time and the initial velocity are found from the system of vector equations[5]

$$0 = x_0 + v_f t_f - \eta t_f^2/2, \quad v_0 = v_f - \eta t_f \tag{3.1}$$

Reducing the first equation to scalar form, we have

$$0 = x_0^2 + 2(x_0, v_f)t_f + t_f^2 - t_f^4/4 \tag{3.2}$$

Eq. (3.2) can have up to three different positive solutions: the smallest of them $t_f = t'_f$ is of interest.

From the second equation of system (3.1), we find the required initial velocity for problem (1.1) $v_0(t'_f) = v'$ which corresponds to a constant control. The velocity which has been found corresponds to the singularities in the graphs of $\phi(\alpha)$ and $t_f(\alpha)$ (see Sections 4 and 5 below respectively).

Note that, when $v_0 = 0$, the angle ϕ^*, which characterizes the control $u(0)$, and the optimal time t_f^* are independent of the angle α. We will now find numerically the initial velocity v_0, $|v_0| \neq 0$ for which $t_f = t_f^*$ for a certain unique α^*. In order to do this, we solve system (1.7) numerically for a known optimal time $t_f = t_f^*$ and a chosen value of the modulus $|v_0| \geq |v'|$. Here, generally speaking, two values α_1 and α_2 are possible, which correspond to the intersection of t_f^* and $t_f(\alpha)$ (Fig. 3). We find the required value $v_0 = v^*$ from the condition that $\alpha_1 = \alpha_2$.

Suppose that, in problem (1.1), the velocity modulus v_0 at the initial point is given and that the angle $\alpha \in [-\pi, \pi]$. The transversality condition

$$a\sqrt{v_1^2 + v_2^2} - b = 0 \tag{3.3}$$

then holds (here v_1 and v_2 are the components of the velocity vector, a and b are unknown constants and, moreover, either $ab < 0$ or $a = b = 0$ (constant control); $v_0 = 0$ when $b = 0$, $a \neq 0$). When $v_0 = 0$, condition (3.3) is satisfied identically and, when $v_0 \neq 0$, the vectors v_0 and η will be collinear. At the same time, at the point corresponding to the time minimum, the vectors η and v_0 are collinear when $|v_0| < |v'|$ and anticollinear when $|v_0| > |v'|$ and, at the maximum point, the vectors η and v_0 are anticollinear when $|v_0| > 0$. In the case when $|v_0| = |v'|$, there is a singularity which leads to a constant control (see Section 4). If the angle α is specified and $|v_0| > 0$, the transversality condition at the initial point has the form

$$a\alpha - b = 0$$

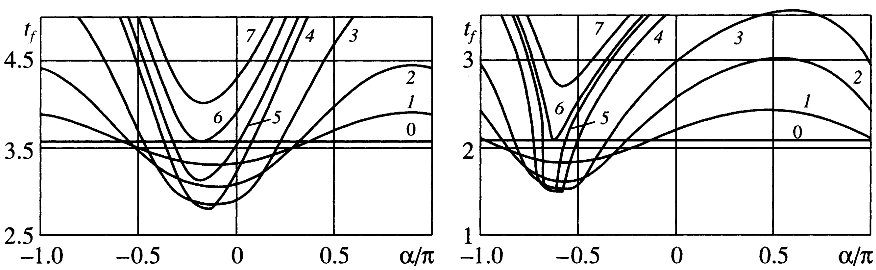

Fig. 3.

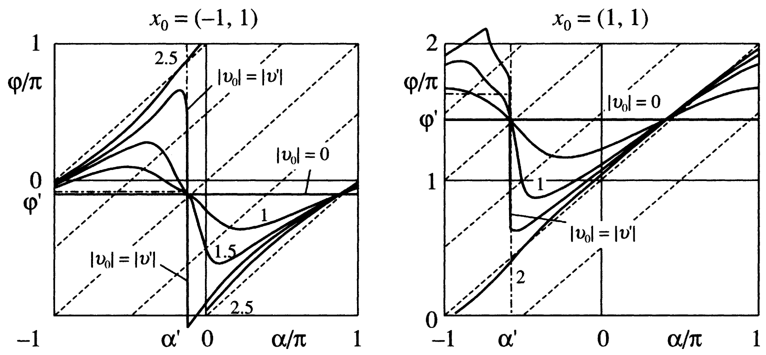

Fig. 4.

where a and b are certain constants and, moreover, when $a=b=0$, a constant control is achieved. Then, either $|v_0|=0$ or the vectors v_0 and η are orthogonal (see Section 5). Note that, if $\alpha=\alpha'$, then there is also a singularity in the case of a constant control (Section 4) which is analogous to the one-dimensional case.

4. Investigation of the control

Graphs of $\phi(\alpha)$ are drawn in Fig. 4 for $x_0=(-1, 1)$ and $x_0=(1, 1)$. These graphs define the angle of the direction of the optimal control with respect to the Ox_1 axis as a function of the direction of the initial velocity, that is, of the angle α for different values of the velocity magnitude $|v|$. The line $\phi=\alpha$ corresponds to conditions of acceleration at the start of the motion and the line $\phi=\alpha-\pi$ to conditions of deceleration. The domain of deceleration is bounded by the lines $\phi=\alpha\pm\pi/2$ and $\phi=\alpha\pm 3\pi/2$. In the case of magnitudes of the initial velocity v_0 which are such that $|v_0|\le|v'|$, the curve $\phi(\alpha)$ intersects the line $\phi=\alpha$. This corresponds to conditions of acceleration, that is, to an increase in the velocity modulus. It follows from the transversality conditions for a problem with an unspecified direction of the initial velocity (Fig. 3) that, at the intersection point, the magnitude of the optimal time $t_f(\alpha)$ has a minimum value.

When $|v_0|>|v'|$, the curve $\phi(\alpha)$ intersects the line $\phi=\alpha+\pi$ at the point of the minimum of $t_f(\alpha)$ at which deceleration conditions hold, that is, at a point where the modulus of the initial velocity decreases.

At the point of intersection of the curve $\phi(\alpha)$ with the line $\phi=\alpha+\pi$, there is a maximum in the time $t_f(\alpha)$, and we have deceleration conditions in the neighbourhood of the initial point when $|v_0|>0$.

Hence, when $|v_0|<|v'|$, acceleration occurs at the point of the time minimum and deceleration occurs when $|v_0|>|v'|$. At the maximum point, there is deceleration when $|v_0|>0$. This property also holds in the case of one-dimensional motion[1] and for the problem considered earlier in Ref. 4.

The transition between the cases of high and low velocities, which occurs when $|v_0|=|v'|$, is shown in Fig. 5 for $x_0=(-1, 1)$ and $x_0=(1, 1)$. The curves correspond to the values of $|v_0|$ as shown in Table 2.

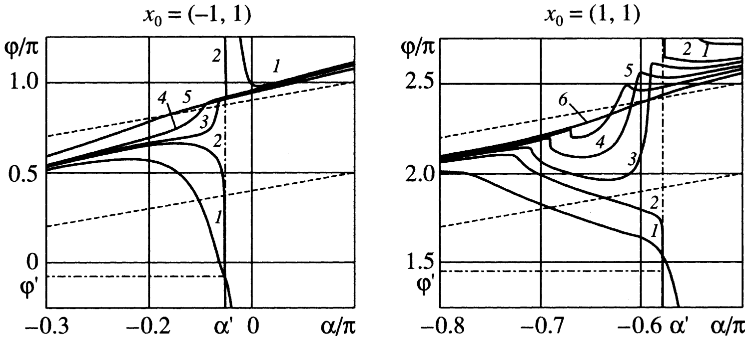

Fig. 5.

Table 2

Curve No.	1	2	3	4	5	6	
$\lvert v_0\rvert$	1.83	$\lvert v'\rvert$	1.86	1.87	1.98		for $x_0 = (-1, 1)$
$\lvert v_0\rvert$	1.25	$\lvert v'\rvert$	1.42	1.46	1.495	1.55	for $x_0 = (1, 1)$

Table 3

Curve No.	1	2	3	4	5	6	7	8	
$\lvert v_0\rvert$	0	0.5	0.9	1.5	$\lvert v'\rvert$	1.988	2.145	2.3	for $x_0 = (-1, 1)$
$\lvert v_0\rvert$	0	0.5	0.7	1	$\lvert v'\rvert$	1.4	1.516	1.6	for $x_0 = (1, 1)$

5. Investigation of the optimal time

Note that, when $|v_0| \leq |v^*|$, a domain exists in which the quantity $t_f(\alpha) \leq t_f^*$ for $x_0 = (-1, 1)$ (the left-hand part of Fig. 4) and for $x_0 = (1, 1)$ (the right-hand part of Fig. 4). The curves correspond to values of $|v_0|$ as shown in Table 3.

This domain is bounded by the lines $t_f = t_f^*$, $\alpha = \alpha_1$, $\alpha = \alpha_2$ (here, the values of the angle α_1 and α_2 correspond to the points of intersection of ϕ^* with the lines $\phi = \alpha \pm \pi/2$ in the graph of $\phi(\alpha)$ with $\alpha_1 + \alpha_2 = \pi$ and with the envelope $\tau(\alpha)$, $\alpha \in [\alpha_1, \alpha_2]$, where τ is the optimal time when the vectors v_0 and η are orthogonal (Section 3). In this envelope, the same optimal time is obtained for a fixed value of the angle α for two different magnitudes of the moduli of the initial velocity. The smallest time when $\alpha = \text{const}$ holds in the case of the orthogonality of the vectors v_0 and η, that is, it corresponds to the point of intersection of $\phi(\alpha)$ with the line $\phi = \alpha \pm \pi/2$. Outside the above-mentioned domain, the minimum time is attained for the condition $|v_0| = 0$.

Hence, in the above-mentioned domain, the optimal time decreases as the velocity increases and it is smallest when $|v_0| = |v'|$.

Interpreting the singularities in the optimal time $t_f(\alpha)$ as properties of optimal trajectories, we conclude that a transition between the types of trajectories when $|v_0| > 0$ corresponds to the collinearity and anticollinearity of the vectors v_0 and η, that is, to the maximum and minimum optimal time.

6. Modification of the results

1°. Consider a problem with a fixed direction of the initial velocity, that is, with an angle α and an arbitrary value of the modulus $|v_0|$. The optimal time t_f and the initial value of the control, which is characterized by the angle ϕ, are found using the corresponding families of curves for the specified vector v_0 (Figs. 3 and 4). If $\alpha \in [\alpha_1, \alpha_2]$, the magnitude of $|v_0|$ is found from the condition for the vectors v_0 and η to be orthogonal. Otherwise, $|v_0| = 0$.

2°. Suppose a problem with a fixed modulus of the initial velocity $|v_0|$ and with an arbitrary direction, characterized by the angle α, is considered. Then, the point of the time minimum is found using the family of curves $t_f(\alpha)$ (Fig. 4) for the specified value of $|v_0|$. The optimal control is found from the graph of $\phi(\alpha)$ (Fig. 3).

3°. We will investigate a more general formulation of the problem with conditions for the start (termination) of the process in the form of inequalities for the quantities $|v_0|$ and α. The initial velocity v_0 is found using the family of curves $t_f(\alpha)$ (Fig. 4) from the condition for a time minimum for the quantities $|v_0|$ and α which satisfy the above-mentioned inequalities. The problem is subsequently investigated in the same way as in cases 1°, 2°.

4°. Suppose the general case of control problems with constraints imposed on the boundary values of $|v_0|$, α and $|v_f|$ holds in the form of inequalities. Then, by a change of coordinate system and normalization of the final velocity, the problem can be successfully reduced to case 3° for different values of the radius vector x_0 in the new system of coordinates.

5°. A problem with constraints, imposed on the value of the velocity and radius vector at the initial and final instants of time, in the form of inequalities can be investigated in a similar manner. For each value of $|x_0|$ and γ (γ is the angle between the x_1-axis and the radius vector x_0), the minimum time is selected by solving problem 4° and the minimum solution is found from the solutions of problem 4° for different x_0.

6°. Suppose the formulation of a control problem with a specified time $t_f = T$ and conditions corresponding to cases 1°–5° is considered. Then, by using the graph of $t_f(\alpha)$ (Fig. 3) for problem (1.1), it is possible to determine the condition for solutions $t_f \leq T$ to exist. If the optimal time t_f turns out to be less than T, it is possible to optimize the magnitude of the control acceleration in a problem with a fixed time of motion, that is, to introduce the function $\tilde{u}(t) = (t_f/T)u(t)$ in order to satisfy the condition $t_f = T$. It is then possible to solve the problem in accordance with case 3°.

The investigation of the optimal time and optimal control carried out above enables us to reveal the basic features of the controlled motion. Considerable attention has been given to the analysis of the optimal time when the modulus of the initial velocity $|v_0|$ and its direction, which is characterized by the angle α, are varied. The corresponding singularities in the optimal time follow from the transversality conditions in the case of a problem with arbitrary values of α and $|v_0|$ respectively (Fig. 3). In particular, for a fixed direction of the initial velocity, there is a minimum time for an angle $\alpha \in [\alpha_1, \alpha_2]$ when the vectors v_0 and η are orthogonal and for values of the angle $\alpha \in [(-\pi, \alpha_1) \cup (\alpha_2, \pi)]$ when $|v_0| = 0$. For a fixed value of the velocity modulus $|v_0|$ when $|v_0| \in (0, |v'|)$, there is a minimum in the optimal time when the vectors v_0 and η are collinear and a maximum when the vectors v_0 and η are anticollinear. For $|v_0| > |v'|$, there is similarly a maximum and minimum when v_0 and η are anticollinear. On the other hand, a change in the types of trajectories occurs at points corresponding to the collinearity and anticollinearity of the vectors v_0 and η. The case of parabolic trajectories, corresponding to a constant optimal control, that is, $|v_0| = |v'|$, is considered separately (Section 3).[5] The singularities in $t_f(\alpha)$ which have been found also hold in the case of three-dimensional motion.

The results obtained enable us to construct a partial synthesis of the optimal control for different values of the current velocity $v = v_0$. The construction of a synthesis, i.e. of a feedback control $u_s(x, v)$, requires additional investigations. An adequate description of the control and motion conditions is the main difficulty. From a computational aspect, the method which has been described enables us to construct and analyse preset motions for arbitrary x_0 and v_0.

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